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by

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ABSTRACT

The status report of which this is an abstract gives in abbreviated form the results of the research done by Tomlinson Fort of Emory University under N.A.S.A. grant.

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First, putting rather weak restrictions on $q_1(z)$, $q_2(z)$, $\ldots$ an intensive study has been made of the series

$$(1) \quad \sum_{n=1}^{\infty} c_n(q_1, q_2, \ldots) \frac{1}{(z + q_1(z))(z + q_2(z)) + \ldots + (z + q_n(z))}$$

A study is then made of difference equation (2). There is a general theory followed by applications of series (1).

$$(2) \quad p_0(z)\, y(z) + p_1(z)y(z + h_1(z)) + \ldots + p_n(z)y(z + h_n(z)) = R(z)$$

where $h_{j+1}(z) = h_1(z) + h_{j-1}(z + h_1(z))$,

$h_j(z) = r_j(z) + s_j(z)i$, $\qquad 0 < r_j < r_{j+1} \longrightarrow \infty$

and $r(z)$ monotonically increasing with x.

Author

Status Report by Tomlinson Fort for period December 1 1963
to June 1 1964 on research done under N.A.S.A. grant made
to Emory University. The title of the project is,

"DIFFERENCE EQUATIONS WITH VARYING DIFFERENCE INTER-

VALS AND DIFFERENCE EQUATIONS AND DIFFERENTIAL

EQUATIONS WITH ALMOST PERIODIC COEFFICIENTS."

1. General

This research has been aimed primarily at the difference
equations (1) below and at the more general system of equations
(2).

Let $h(z)$ be a function to be restricted somewhat as the
work demands. We also define a sequence of functions as follows:

$$h_1(z) = h(z)$$
$$h_2(z) = h(z) + h(z + h(z))$$
$$h_3(z) = h(z) + h(z + h(z)) + h(z + h(z) + h(z + h(z)))$$
$$\circ \ \bullet \ \circ \ \bullet \ \circ \ \circ \ \bullet \ \circ \ \bullet \ \circ \ \circ \ \circ$$
$$h_n(z) = h(z) + h_{n-1}(z + h(z)).$$

We now write the equation

$$(1) \quad p_0(z)y(z) + p_1(z)y(z + h_1(z)) + p_2(z)y(z + h_2(z)) + \ldots$$
$$+ p_n(z)y(z + h_n(z)) = R(z).$$

We assume $p_n(z) \neq 0$ and $p_0(z) \not\equiv 0$ in the portion of the plane considered. We note that to get $y(z + h_j(z))$ we replace z by $z + h(z)$ in $y(z + h_{j-1}(z))$.

The linear system above referred to is

$$y_1(z + h(z)) = a_{11}(z)y_1(z) + a_{12}(z)y_2(z) + \ldots + a_{1n}(z)y_n(z) + r_1(z)$$
$$y_2(z + h(z)) = a_{21}(z)y_1(z) + a_{22}(z)y_2(z) + \ldots + a_{2n}(z)y_n(z) + r_2(z)$$
$$\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet$$
$$y_n(z + h(z)) = a_{n1}(z)y_1(z) + a_{n2}(z)y_2(z) + \ldots + a_{nn}(z)y_n(z) + r_n(z)$$

or in matrix form

$$(2) \qquad Y(z + h(z)) = A(z)\, Y(z) + R(z).$$

I have done a great deal of work and have made considerable progress toward building a satisfactory theory of equation (1). So far as I know this is a new field. The last word on it probably never will be written. In the report which follows theorems are stated. Usually, for the sake of brevity, proofs are not given. I understand that brevity is desired by the management of NASA. I have found some of the theorems difficult and have taken a great deal of time in their discovery and proof.

It is planned to submit the material in this report with detailed proofs to one of the American journals of mathematics.

First! In order to treat equation (1) it is necessary to develop proper tools. The most useful tool that I have found is (5) below. A most useful special form of (5) is (3). These series are of interest in themselves and much of this report is given to their study.

$$(3) \quad a_1 \frac{1}{z + q_1(z)} + a_2 \frac{q_1(z)}{(z + q_1(z))(z + q_2(z))}$$

$$+ a_3 \frac{q_1(z)\, q_2(z)}{(z + q_1(z))(z + q_2(z))(z + q_3(z))} + \ldots\ldots$$

$$+ a_n \frac{q_1(z)\, q_2(z) \ldots\ldots q_{n-1}(z)}{(z + q_1(z))(z + q_2(z))\ldots(z + q_n(z))} + \ldots\ldots$$

We let $q_n(z)\, q_{n-1}(z) \ldots q_{n-j+1}(z) = (q_n(z))^{(j)}$,

$(z + q_n(z))(z + q_{n-1}(z))\ldots(z + q_{n-j+1}(z)) = (z + q_n(z))^{(j)}$

Similar notation is used in other circumstances. It is believed that the context will always make clear the nature of the notation. In particular, $0^{(j)} = 1$, even $0^{(0)} = 1$.

We now write in place of (3)

$$(4) \quad \sum_{n=1}^{\infty} a_n \frac{(q_{n-1}(z))^{(n-1)}}{(z + q_n(z))^{(n)}}$$

We shall for the present concentrate on (5) below.

$$(5) \quad \sum_{n=1}^{\infty} c_n(q_1, q_2, \ldots) \frac{1}{(z + q_n(z))^{(n)}}$$

In order for (5) to be meaningful it is, of course, necessary to restrict $c_n(q_1, q_2, \ldots)$. We shall write this $c_n(q)$ and adopt similar notation with other functions.

We shall frequently have occasion to refer to

$$(6) \quad \sum_{n=1}^{\infty} c_n(q(z)) \frac{1}{(z_0 + q_n(z))^{(n)}}$$

We let $z = x + yi$ where x and y are real and i the imaginary unit. We also let $q_j(z) = r_j(z) + s_j(z) i$ where $r_j(z)$ and $s_j(z)$ are real. We require that $r_j(z) > 0$ and that $r_j(z) \to \infty$ when $j \to \infty$, and we sometimes require that $s_j(z)/r_j(z) \to 0$ uniformly over the region considered, and that $r_j(z)$ be monotonically increasing in x.

There is a relation between $h_n(z)$ and $h_{n-1}(z)$ of equation (1). This relation does not necessarily exist between $q_n(z)$ and $q_{n-1}(z)$. To emphasize this point the letter "h" was changed to "q". In section 3 we return to the "h" of equation (1).

2. Series (5)

Let $b_n(z_0 z) = \dfrac{(z_0 + q_1(z_0))(z_0 + q_2(z_0)\ldots(z_0 + q_n(z_0))}{(z + q_1(z))(z + q_2(z))\ldots(z + q_n(z))}$.

Theorem 1. Let $x_0 \geq 0$. In case $\displaystyle\sum_{j=1}^{\infty} (y_0 + s_j(z))^2/(x_0 + r_j(z))^2$

converges uniformly to a bounded function over the half-plane

$x \geq x_0$ then

(i) $b_n(z_0 z)$ remains uniformly bounded over the half-plane

$x \geq x_0$ and

(ii) given any two positive constants a and b and positive

integer m there exists a corresponding number N, independent

of z such that over the half plane $x \geq x_0$

$$\left| b_n(z, z_0) \right| < N/[a(x - x_0)^m + b].$$

Theorem 2. Let $\triangle b_n = b_{n+1} - b_n$, also let a and b be positive

constants and m a positive integer. If $\displaystyle\sum_{n=1}^{\infty} [y_0 + \sigma_n(z)]^2/[x_0 + r_n(z)]^2$

converges uniformly over the half-plane $x \geq x_0$ to a bounded function

then $\displaystyle\sum_{n=1}^{\infty} \left| \triangle b_n(z_0 z) \right|$ is uniformly bounded over the sectorial

region defined by $\left| y - y_0 \right| \leq [a(x - x_0)^m + b](x - x_0)$ and $x \geq x_0$

Theorem 3. If $x_0 > 0$ then $\displaystyle\sum_{j=1}^{\infty} \left| \triangle b_j(z_0, z_0) \right|$ remains uni-

formly bounded over the sector defined by $x > x_0$ and $\left| y - y_0 \right| /x - x_0$

$\leq \tan \theta$ where θ is a fixed positive angle less than $\Pi/2$.

A region from which circular neighborhoods, with positive minimum radius, of all those points at which $z + q_j(z) = 0$ for any j are removed is called a deleted region.

Theorem 4. If (6) converges uniformly over the deleted sectorial region, R, defined by $x \geq x_0$ and $\left| y - y_0 \right| /(x - x_0) < \tan \theta$ where θ is any fixed positive angle then (5) converges uniformly over R also.

Theorem 5. If $\displaystyle\sum_{j=1}^{\infty} [y_0 + s_j(z)]^2/[x_0 + r_j(z)]^2$ converges uniformly to a bounded function and if (6) converges uniformly over the deleted region S defined by $x \geq x_0$ and $\left| y - y_0 \right| \leq [a(x - x_0)^m + b](x - x_0)$ where a and b are any positive constants and m any positive integer then (5) converges uniformly over S also.

Theorem 6. If (5) converges absolutely at $z = z_0$ and if $\displaystyle\sum_{n=1}^{\infty} [y_0 + s_n(z)]^2/[x_0 + r_n(z)]^2$ converges uniformly to a bounded function and if $\dfrac{c_n(q(z_0))}{c_n(q(z))} < M$ then (5) converges absolutely uniformly over the half-plane defined by $x \geq x_0$

Theorem 7. If (6) converges absolutely at z_0 then (5) converges absolutely uniformly over the deleted sectorial region defined by $x \geq x_0$ and $|y - y_0| / x - x_0 \leq \tan \theta$ where $0 < \theta < \Pi/2$.

Theorem 8. If m is a positive integer and if $\sum_{j=1}^{\infty} q_j^{-s}$

where $s \geq m + 1$ converges uniformly over the half-plane $x \geq x_0$ and diverges when $x \leq m$ then (3) converges at the

same points as does $\sum_{n=1}^{\infty} c_n e^{gn}$ where $g_n = \sum_{j=1}^{m} (-1)^j \frac{z^j}{j} \sum_{t=1}^{j} q_j^{-t}$.

The two series also converge absolutely at the same points.

In case the q_j's are constants and $m = 1$ then $\sum_{n=1}^{\infty} c_n e^{gm}$ is a Dirichlet series.

3. Transformations of Series

Series of type (5) are, of course, subject to classical theorems on absolute and uniform convergence.

In this section we shall assume absolute convergence and than when z is real h(z) is real and positive and $h_{j+1} - h_j \geq 0$ also that $\sum_{n=1}^{\infty} \frac{1}{h_n(z)}$ is divergent.

(a) Step-up theorems

Theorem 9. Let $k > 1$ be an integer. Let

$$(7) \quad F(z) = \sum_{n=0}^{\infty} c_{n+1}(q) \frac{1}{(z + h_{n+1}(z))^{(n+1)}} \, ,$$

then also

$$(8) \quad F(z) = \sum_{n=0}^{\infty} \frac{{}_k B_n}{(z + h_{n+k})^{(n+1)}} \, ,$$

where

$$_2 B_n = \sum_{j=0}^{n} c_{j+1}(h(z))(h_{n+1}(z) - h_1(z))$$

and in general

$$_k B_n = \sum_{j=0}^{n} {}_{k-1} B_j \, [h_{n+k-1} - h_{k-1}]^{(n-j)}$$

We thus have additional representations for $F(z)$.

Notice the return to the letter "h" which is subject to the definition written at the beginning of this report.

(b) Step-down theorems

Theorem 10. If $F(z)$ is given by (8) then it is also given by (7).

Theorem 11. If

$$F(z) = \sum_{n=0}^{\infty} c_{n+1}(h) \frac{1}{(z + h_{n+k})^{(n+1)}}$$

then also

$$F(z) = \sum_{n=1}^{\infty} \frac{c_{n-1}(h)(h_{k-1} - h_{k+n-2}) + c_n(h)}{(z + h_{n+k-2})^{(n)}}$$

This step-down process can be repeated.

(c) Multiplication

The discovery and proof of a satisfactory multiplication theorem was the most difficult thing in this research.

Theorem 12.   Let

$$F_1 = \sum_{n=0}^{\infty} a_{n+1}(h) \frac{1}{(z + h_{n+1}(z))^{(n+1)}}$$

and

$$F_2 = \sum_{n=0}^{\infty} b_{n+1}(h) \frac{1}{(z + h_{n+1}(z))^{(n+1)}} ,$$

then

$$F_1 F_2 = \sum_{n=1}^{\infty} c_n(h) \frac{1}{(z + h)^{(n)}}$$

where

$$c_n = \sum_{j=1}^{n} b_j \, {}_j B_n \qquad \text{[for } {}_j B_n, \text{ see Theorem 9]}$$

It is to be noticed that ${}_j B_n$ depends upon a_1, a_2, ..., a_n, ... and h, not the b's and not z except through h and the a's.

If the a's and h are constants with reference to z then so
are the c's. Our theorem then states that the product
of two functions represented by series of type (5) can be
represented by another series of the same type. In particular
that the product of two functions given by factorial series
can be represented by a factorial series, a known result.

4. Uniqueness Theorem

Theorem 13. If $\displaystyle\sum_{n=0}^{\infty} \frac{a_n(h)}{(z + h_n(z))^{(n+1)}}$ converges uniformly

to zero over the x-axis when $x \geq x_0$ and if $\dfrac{a_n(h)}{(x + h_n(z))^{(n+1)}} \to 0$

when $x \longrightarrow \infty$ then $a_n(h) \equiv 0$.

5. General Existence Theorem

Theorem 14. Let $h(z) = h(x)$ be real and continuous, positive
and monotonically increasing in x. If we choose any real number,
c, and in the complex plane draw the lines, $x = c$, $x = c + h_1(c)$,
$x = c + h_2(c) \ldots x = c + h_n(c) + \ldots$. Then in the bands
$c \leq x < c + h_1(c)$, $c + h_1(c) \leq x < c + h_2(c), \ldots c + h_{n-1}(c)$
$\leq x < c + h_n(c)$ assign y arbitrarily. Then, provided the
coefficients in (1) are defined when $x \geq c$, there exists one

and only one solution of (1) in the half-plane x $\geq$ c, which
has the values just assigned in the n bands as above de-
termined.

As a matter of fact, the requirement that h(z) be real
and positive is unnecessarily restrictive. However, it does
give us a picture of bands similar to the picture usually
considered when h $\equiv$ 1, which is the case in the ordinary
difference equation.

6. A Particular Solution of the Non-homogeneous Equation

The series which we have discussed in this report are
of interest in themselves and doubtless have many interesting
applications. We proceed to apply them to the non-homogeneous
difference equation with varying difference intervals as ex-
plained in section 1 of this paper.

Consider

(9) $p_0(z)y(z) + p_1(z)y(z + h_1(z)) + p_2(z)y(z + h_2(z)) + \ldots +$
$$+ p_n(z)y(z + h_n(z)) = R(z).$$

We assume that each of the coefficients p_0, $\ldots$, p_n and R(z)
are expressed in form (1) below

$$(10) \quad a_{-j}(h)(z + h_j)^{(j)} + a_{-(j-1)}(h)(z + h_{j-1})^{(j-1)} + \ldots$$

$$+ a_{-1}(h)(z + h_1) + a_0(h) + \sum_{n=1}^{\infty} a_n(h) \frac{1}{(z + h)^n} \, ,$$

where $a_{-j}, \ldots, a_0$ are bounded in z. We assume, moreover, that the series are absolutely convergent when $x \geq x_0$.

We assume $R \neq 0$ and $p_n(z) \neq 0$ at any point when $x \geq x_0$.

Theorem 15. Equation (9) has a unique solution expressible in the form (10) when $x \geq x_0$.

The general idea of the proof is to assume y an absolutely convergent series of the form (10), apply step-up and step-down and multiplication theorems. Then apply the method of undetermined coefficients consistent with our uniqueness theorem. We obtain a formal solution, to show that this formal solution converges absolutely we set up a majorant series as is frequently done in analogous cases in classical mathematics.

If this process is applied to the homogeneous equation, we get $y = 0$.

7. General Periodic Functions

A function $P(z)$ will be called general periodic if when

$x \geq x_0$

$P)z) = P(z + h_1(z)) = P(z + h_2(z)) = P(z + h_3(z)) = \ldots$

It seems that such functions will have a large theory. I have not developed this as yet.

8. General Theory of the Homogeneous Linear Equation

We consider equation (1) with $R = 0$, namely

(11) $p_0(z)y(z) + p_1(z)y(z + h_1(z)) + \ldots + p_n(z)y(z + h_n(z)) = 0$

It is immediate that any general periodic function is a solution of (11) if

$$p_0(z) + p_1(z) + \ldots + p_n(z) = 0.$$

It is also immediate that if $y(z)$ is any solution of (11) and $v(z)$ any general periodic function then $v(z)y(z)$ is also a solution of (11).

If we apply our general existence theorem to (11) we can exhibit as many solutions as we wish. We find that the theorem of Casorati for ordinary difference equations goes over. First we call functions, $u_1(z), \ldots, u_j(z)$ linearly dependent if there exist j general periodic functions $v_1(z), \ldots, v_j(z)$ such that

$$v_1(z)u_1(z) + v_2(z)u_2(z) + \ldots + v_j(z)u_j(z) = 0$$

over the portion of the complex plane considered.

9. The Gamma Function

If

$$\lim_{b \to \infty} [\sum_{t=x}^{b} \log t - \int_{1}^{b} \log t \, dt] \quad \text{exists where}$$

$x \geq 1$ and where summation is taken for the values $t = x$,

$x + h_1(z)$, $x + h_2(x)$, $\ldots$, $x + h_n(x) = b$, then there exists

a function not identically zero satisfying the relation (12)

$$(12) \qquad\qquad y(x + h(x)) = x \, y(x).$$

This function can be analytically continued over the half-

plane $x > 0$. If we replace $h(x)$ by $h(z)$ where $h(z)$ is

analytic and $h(z)$ real when z is real, then the analytic

function of z will satisfy (12).

If $y(z)$ satisfies (12) then so does $v(z)y(z)$ where

$v(z)$ is general periodic.

10. Analytic Solutions of the Homogeneous Equation

If the coefficients of (11) are analytic over some re-

gion, R, if $h(z)$ is analytic over R it is desirable to prove

the existence of analytic solutions and to get a series or

other analytic representation for such functions. Even if

all initial values belong to one analytic function, our

general existence theorem will not, in general, yield an analytic function.

Progress has been made on this problem. If the coefficients $p_0(z)$, ..., $p_n(z)$ are given by series of the form (10) then a necessary form for a fundamental system of solutions has been found. I am confident that further work and perhaps further restrictions will prove that these series really are solutions.

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